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Modal control of vibration in rotating machines and other generally damped systems

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Abstract

Second-order matrix equations arise in the description of real dynamical systems. Traditional modal control approaches utilise the eigenvectors of the undamped system to diagonalise the system matrices. A regrettable consequence of this approach is the discarding of residual off-diagonal terms in the modal damping matrix. This has particular importance for systems containing skew-symmetry in the damping matrix which is entirely discarded in the modal damping matrix. In this paper, a method to utilise modal control using the decoupled second-order matrix equations involving non-classical damping is proposed. An example of modal control successfully applied to a rotating system is presented in which the system damping matrix contains skew-symmetric components.

1. Introduction

Traditional control approaches, such as pole placement methods [1], deal with the physical system in first-order state-space form. The ambitions of this paper are to control the physical system in second-order form. Very little literature is available in regards to direct second-order control, see for example Ref. [2]. Many obvious advantages over first-order control are available: (1) Physical insight of the system is preserved. (2) Computational efficiency, since the dimension of the second-order system is smaller than that of the state-space form. (3) Symmetry and structure of the systems can be preserved where desired.

Many structural and dynamic systems are described by the second-order equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{S}\mathbf{f}_{\text{phy}}(t), \tag{1}$$

where $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices, respectively, $\mathbf{q}(t) \in \mathbb{R}^n$ the vector of physical coordinates, $\mathbf{f}_{\text{phy}}(t) \in \mathbb{R}^r$ the vector of applied forces and $\mathbf{S} \in \mathbb{R}^{n\times r}$ is a selection matrix determining the locations of applied forces. For the sake of brevity this paper assumes that forces are available at all locations such that $r = n$ resulting in $\mathbf{S}$ being equal to an $n \times n$ identity matrix.

Modal control is a particular control method in which the physical response of a system is divided into modes associated with corresponding natural frequencies. A standard control approach is to move the system

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eigenvalues into a stable region. The essence of modal control is that since the eigenvectors of a system do not contribute to the asymptotic stability of a system then any effort expended on altering them represents wasted effort. This is the control approach utilised in this paper.

Meirovitch and Baruh introduced a first-order modal control method using a state-space representation of system containing skew-symmetry in the damping matrix [3]. The modal contributions are extracted from the physical quantities using modal filters [4] but the method does not derive an inverse modal filter to revert the modal quantities back to the physical domain. A backward transformation is defined which allows only one half of the modelled modes to be controlled. Meirovitch and Baruh proposed to control only the lower-order modelled modes with justification for this being that the higher-order modes are more difficult to excite hence do not contribute significantly to the system response. The method proposed in this paper removes this constraint by defining an inverse filter making it possible to control all the modelled modes.

Modern computers have enough computational capacity such that worries concerning the expansion of the control problem to $2n$ rather than an n-dimensional problem is not an issue for moderate values of n. However, redefining the second-order equations of motion into a first-order realisation has the disadvantage of destroying some properties such as symmetry and definiteness of the matrices describing the motion [5]. Here, direct second-order techniques allow the retention of the natural form of dynamic systems arising from Newtonian mechanics.

Traditional modal control for second-order systems such as the 'Independent Modal Space Control' (IMSC) method used by Baz et al. [6] utilise the mass normalised left and right eigenvectors, $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$, of the undamped system to diagonalise the system matrices. Although, the method outlined is developed for self-adjoint systems one may realise that the same method is applicable when this criteria is relaxed the difference being that the left and right eigenvectors are distinct. Thus, for the non-self-adjoint one uses the distinct left and right eigenvectors of the undamped system to attempt the diagonalisation process. For the self-adjoint case one finds that $\mathbf{\Phi}_L = \mathbf{\Phi}_R$. The coordinate transformation $\mathbf{q}(t) = \mathbf{\Phi}_R \mathbf{q}_m(t)$ is applied and the system matrices pre-multiplied by the transpose of the left eigenvectors, $\mathbf{\Phi}_L^{\mathrm{T}}$

From

$$\mathbf{\Phi}_L^{\mathrm{T}} \mathbf{M} \mathbf{\Phi}_R \ddot{\mathbf{q}}_m + \mathbf{\Phi}_L^{\mathrm{T}} \mathbf{D} \mathbf{\Phi}_R \dot{\mathbf{q}}_m + \mathbf{\Phi}_L^{\mathrm{T}} \mathbf{K} \mathbf{\Phi}_R \mathbf{q}_m = \mathbf{\Phi}_L^{\mathrm{T}} \mathbf{f}_{\mathrm{phy}}, \tag{2}$$

one has

$$\mathbf{I} \ddot{\mathbf{q}}_m + \mathbf{\Gamma} \dot{\mathbf{q}}_m + \mathbf{\Lambda}^2 \mathbf{q}_m = \mathbf{\Phi}_L^{\mathrm{T}} \mathbf{f}_{\mathrm{phy}}, \tag{3}$$

where $\mathbf{q}_m(t)$ represents the modal coordinates of the system. For ease of reading the notation indicating the dependence on time is removed.

The new damping matrix $\mathbf{\Gamma}$ is assumed diagonal with any remaining off-diagonal terms in the modal damping matrix traditionally discarded [7]. However, for rotating systems involving substantial gyroscopic terms stripping the off-diagonal terms in the damping matrix is in effect ignoring the gyroscopic terms themselves. Thus, it is proposed here to use the 'Structure Preserving Transformations' (SPTs) developed by Garvey et al. [8,9] to diagonalise the second-order system matrices and decouple the system equations of motion without need to discard any terms involved in the description of the system.

2. Structure preserving transformations

The notion of the 'Lancaster Augmented Matrices' (LAMs) is introduced here. For a second-order system there exists three LAMs which can be produced by inspection to be,

$$\underline{\mathbf{A}_0} = \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix}, \quad \underline{\mathbf{A}_1} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix}, \quad \underline{\mathbf{A}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix}. \tag{4}$$

The LAMs allow the second-order system to be represented in a reduced form

$$\underline{\mathbf{A}_k}\, \underline{\mathbf{q}_A} - \underline{\mathbf{A}_{k-1}}\, \underline{\dot{\mathbf{q}}_A} = \underline{\mathbf{f}_{Ak}}, \quad k = 1, 2. \tag{5}$$

The vectors $\underline{\mathbf{q}_A}$ and $\underline{\mathbf{f}_{Ak}}$ may be defined

$$\underline{\mathbf{q}_A} := \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \quad \underline{\mathbf{f}_{A1}} := \begin{bmatrix} \mathbf{f}_{\text{phy}} \\ \mathbf{0} \end{bmatrix}, \quad \underline{\mathbf{f}_{A2}} := \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{\text{phy}} \end{bmatrix}. \tag{6}$$

For the purposes of this paper a 'Structure Preserving Transformation' (SPT) can be considered to be a coordinate transformation applied to the LAMs of one system such that the LAMs of a new system are obtained. As such, these SPTs constitute a left and right $2n \times 2n$ transformation matrix, $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$, respectively, according to

$$\underline{\mathbf{T}_L^{\mathrm{T}}}\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R} = \underline{\mathbf{B}_k} \quad \forall k = 0, 1, 2. \tag{7}$$

Thus the new LAMs are represented by $\underline{\mathbf{B}_k}$ containing the new second-order system matrices $\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1$. The structure of the transformation matrices can be shown to have the following form:

$$\underline{\mathbf{T}_L} = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\mathbf{G}_L\mathbf{D}_A^{\mathrm{T}} & -\mathbf{G}_L\mathbf{M}_A^{\mathrm{T}} \\ \mathbf{G}_L\mathbf{K}_A^{\mathrm{T}} & \mathbf{F}_L + \frac{1}{2}\mathbf{G}_L\mathbf{D}_A^{\mathrm{T}} \end{bmatrix}^{-1}, \quad \underline{\mathbf{T}_R} = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\mathbf{G}_R\mathbf{D}_A & -\mathbf{G}_R\mathbf{M}_A \\ \mathbf{G}_R\mathbf{K}_A & \mathbf{F}_R + \frac{1}{2}\mathbf{G}_R\mathbf{D}_A \end{bmatrix}^{-1}, \tag{8}$$

where $\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R \in \mathbb{R}^{n\times n}$ are arbitrary pre-defined matrices subject to the necessary constraint

$$\mathbf{F}_R\mathbf{G}_L^{\mathrm{T}} + \mathbf{G}_R\mathbf{F}_L^{\mathrm{T}} = 0. \tag{9}$$

3. Diagonalising structural preserving transformations

It is desired to decouple the original equations of motion such that the new system matrices $\mathbf{K}_B, \mathbf{D}_B$ and $\mathbf{M}_B$ are diagonal. For non-defective systems [10] it is always possible to choose a non-unique SPT such that the entries in the new LAMs become diagonal. Such an SPT is referred to as a 'diagonalising SPT' (DSPT) and a 4-step process of calculating the DSPT is presented here.

1. Calculate the left ($\underline{\mathbf{\Psi}_L}$) and right ($\underline{\mathbf{\Psi}_R}$) eigenvectors of reduced system

$$\underline{\mathbf{A}_1} - \tau\,\underline{\mathbf{A}_0}, \tag{10}$$

   where $\tau \equiv \mathrm{d}/\mathrm{d}t$.
2. Calculate the n monic 'single degree of freedom' (sdof) systems corresponding to conjugate eigenvalue pairs, $\lambda_{j(1,2)} = \alpha \pm \mathrm{i}\beta$, found in step 1. For systems with real pairs of roots the same method applies through appropriate pairing

$$d_j = \lambda_{j1} + \lambda_{j2}, \quad k_j = \frac{(\lambda_{j2} + \lambda_{j1})^2 - (\lambda_{j2} - \lambda_{j1})^2}{4}, \quad m_j = 1, \ j = 1, \ldots, n. \tag{11}$$

3. Knowing the new diagonal system matrices form the new LAMs $\underline{\mathbf{B}_0}$ and $\underline{\mathbf{B}_1}$ representing the new diagonal system and calculate their corresponding left ($\underline{\mathbf{\Theta}_L}$) and right ($\underline{\mathbf{\Theta}_R}$) eigenvectors.
4. Since the two reduced systems have identical Jordan form, appropriate scaling of the eigenvectors yields the following equalities:

$$\underline{\mathbf{\Psi}_L^{\mathrm{T}}}\,\underline{\mathbf{A}_1}\,\underline{\mathbf{\Psi}_R} = \underline{\mathbf{\Lambda}} = \underline{\mathbf{\Theta}_L^{\mathrm{T}}}\,\underline{\mathbf{B}_1}\,\underline{\mathbf{\Theta}_R} \quad \text{and} \quad \underline{\mathbf{\Psi}_L^{\mathrm{T}}}\,\underline{\mathbf{A}_0}\,\underline{\mathbf{\Psi}_R} = \underline{\mathbf{I}} = \underline{\mathbf{\Theta}_L^{\mathrm{T}}}\,\underline{\mathbf{B}_0}\,\underline{\mathbf{\Theta}_R}, \tag{12}$$

   where $\underline{\mathbf{\Lambda}}$ is the diagonal matrix of corresponding eigenvalues and $\underline{\mathbf{I}}$ is the identity matrix. From Eq. (12) one may recognise that

$$(\underline{\mathbf{\Theta}_L^{-\mathrm{T}}}\,\underline{\mathbf{\Psi}_L^{\mathrm{T}}})\,\underline{\mathbf{A}_k}(\underline{\mathbf{\Psi}_R}\,\underline{\mathbf{\Theta}_R}^{-1}) = \underline{\mathbf{B}_k}, \quad k = 0, 1, \tag{13}$$

thus $\underline{\mathbf{T}_R} = (\underline{\mathbf{\Psi}_R}\,\underline{\mathbf{\Theta}_R^{-1}})$ and $\underline{\mathbf{T}_L} = (\underline{\mathbf{\Psi}_L}\,\underline{\mathbf{\Theta}_L^{-1}})$.

It may be noted that the above process for finding the diagonalising SPT only requires one eigenvalue problem. The eigenvectors of the diagonal LAMs, $\underline{\boldsymbol{\Theta}_L}$ and $\underline{\boldsymbol{\Theta}_R}$, have a sparse form such that their calculation is trivial.

4. Modal filters

The ambition of this paper is to develop a new second-order modal control technique. Therefore, the necessary question arises of how to extract the second-order modal contributions from the state-space system. The derivation of the modal filters for SPT-based control is presented here. For the purpose of this section the Laplace domain is favoured rather than the time domain and the Laplace variable s is introduced.

With the partitioning

$$\underline{\mathbf{q}_A} =: \begin{bmatrix} \underline{\mathbf{q}_{A1}} \\ \underline{\mathbf{q}_{A2}} \end{bmatrix}, \quad \underline{\mathbf{f}_{A1}} =: \begin{bmatrix} \underline{\mathbf{f}_{A1,1}} \\ \underline{\mathbf{f}_{A1,2}} \end{bmatrix}, \quad \underline{\mathbf{f}_{A2}} =: \begin{bmatrix} \underline{\mathbf{f}_{A2,1}} \\ \underline{\mathbf{f}_{A2,2}} \end{bmatrix}, \tag{14}$$

it is possible to extract a definition of the original second-order system from the state-space representation

$$\mathbf{q}(s) = \underline{\mathbf{q}_{A1}}, \quad \mathbf{f}_{\text{phy}}(s) = \underline{\mathbf{f}_{A1,1}} + s\,\underline{\mathbf{f}_{A1,2}}. \tag{15}$$

Eq. (15) has been generalised such that it is assumed that the forcing part of the state-space representation, $\underline{\mathbf{f}_{A1}}$, contains non-zeros. Whilst for the original system this is clearly not the case, the definition allows the extension to the transformed problem for which $\underline{\mathbf{f}_{B1}}$ is generally fully populated. Eq. (15) can be proved mechanistically. Expanding Eq. (5) for $k = 1$ and 2 yields

$$\mathbf{K}_A(\underline{\mathbf{q}_{A2}} - s\,\underline{\mathbf{q}_{A1}}) = \underline{\mathbf{f}_{A2,1}}, \tag{16}$$

$$\mathbf{D}_A(\underline{\mathbf{q}_{A2}} - s\,\underline{\mathbf{q}_{A1}}) = \underline{\mathbf{f}_{A2,2}} - \underline{\mathbf{f}_{A1,1}}, \tag{17}$$

$$\mathbf{M}_A(\underline{\mathbf{q}_{A2}} - s\,\underline{\mathbf{q}_{A1}}) = -\,\underline{\mathbf{f}_{A1,2}}. \tag{18}$$

Substituting $\underline{\mathbf{f}_{A2,2}}$ from Eq. (5) into Eq. (17) and subtracting s multiplied by Eq. (18) yields

$$(\mathbf{K}_A + s\mathbf{D}_A + s^2\mathbf{M}_A)\,\underline{\mathbf{q}_{A1}} = \underline{\mathbf{f}_{A1,1}} + s\,\underline{\mathbf{f}_{A1,2}}. \tag{19}$$

Hence Eq. (15) is proved. It is prudent at this juncture to point out that following similar methodology the equations of motion may also be represented in terms of $\underline{\mathbf{q}_{A2}}$.

By applying the SPTs one has the new transformed equations of motion

$$\underline{\mathbf{B}_k}\,\underline{\mathbf{q}_B} - \underline{\mathbf{B}_{k-1}}\,s\,\underline{\mathbf{q}_B} = \underline{\mathbf{T}_L^{\mathrm{T}}}\,\underline{\mathbf{f}_{Ak}} = \underline{\mathbf{f}_{Bk}}. \tag{20}$$

Thus, one may extract the new second-order system of equations in terms of partitions of the new state variable $\underline{\mathbf{q}_B}$

$$(\mathbf{K}_B + s\mathbf{D}_B + s^2\mathbf{M}_B)\mathbf{q}_B(s) = \mathbf{f}_B(s). \tag{21}$$

From Eq. (15) the following are defined:

$$\mathbf{q}_B(s) = \underline{\mathbf{q}_{B1}}, \quad \mathbf{f}_B(s) = \underline{\mathbf{f}_{B1,1}} + s\,\underline{\mathbf{f}_{B1,2}}. \tag{22}$$

The obvious question now arises, what is the relationship between the old and new coordinate sets? Acknowledging that the system coordinates are transformed using the definition $\underline{\mathbf{q}_B} = \underline{\mathbf{S}_R}\,\underline{\mathbf{q}_A}$ where $\underline{\mathbf{S}_R} = \underline{\mathbf{T}_R^{-1}}$ one has

$$\begin{bmatrix} \underline{\mathbf{q}_{B1}} \\ \underline{\mathbf{q}_{B2}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{R11} & \mathbf{S}_{R12} \\ \mathbf{S}_{R21} & \mathbf{S}_{R22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{A1}} \\ \underline{\mathbf{q}_{A2}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{R11}\,\underline{\mathbf{q}_{A1}} + \mathbf{S}_{R12}\,\underline{\mathbf{q}_{A2}} \\ \mathbf{S}_{R21}\,\underline{\mathbf{q}_{A1}} + \mathbf{S}_{R22}\,\underline{\mathbf{q}_{A2}} \end{bmatrix}. \tag{23}$$

The definition of $\mathbf{q}_B$ and $s\mathbf{q}_B$ from Eq. (22) is used to see that the new coordinate set is related to the old through the definition

$$\mathbf{q}_B = [\mathbf{I} \quad \mathbf{0}]\,\underline{\mathbf{T}_R^{-1}}\,\underline{\mathbf{q}_A}. \tag{24}$$

This result allows the introduction of a "right" filter of the form

$$\mathbf{q}_B = \mathbf{U}_{01}\mathbf{q} + s\mathbf{U}_{11}\mathbf{q}, \tag{25}$$

where $\mathbf{U}_{i,j} = \mathbf{S}_{Rj,i+1}$ and where $\mathbf{q}_B$ represents the modal displacement obtained through the right filter. The modal displacement is determined through knowledge of physical displacements and velocities. Accordingly the modal velocity may be obtained with knowledge of the physical accelerations to be

$$s\mathbf{q}_B = s\mathbf{U}_{01}\mathbf{q} + s^2\mathbf{U}_{11}\mathbf{q}. \tag{26}$$

It is now necessary to introduce the left filter to allow the relationship between new and old forcing vectors to be established.

From the result of Eq. (20) it is clear that the vector $\underline{\mathbf{f}_{B1}} = \underline{T_L^{\mathrm{T}}}\,\underline{\mathbf{f}_{A1}}$. Knowing from definition given in Eq. (6) that $\underline{\mathbf{f}_{A1,2}} = 0$ it can be deduced that

$$\begin{bmatrix} \underline{\mathbf{f}_{B1,1}} \\ \underline{\mathbf{f}_{B1,2}} \end{bmatrix} = \begin{bmatrix} \underline{T_{L,11}^{\mathrm{T}}} & \underline{T_{L,21}^{\mathrm{T}}} \\ \underline{T_{L,12}^{\mathrm{T}}} & \underline{T_{L,22}^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{f}_{A1,1}} \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{T_{L,11}^{\mathrm{T}}} & \underline{\mathbf{f}_{A1,1}} \\ \underline{T_{L,12}^{\mathrm{T}}} & \underline{\mathbf{f}_{A1,1}} \end{bmatrix}. \tag{27}$$

Thus one may define the left filter

$$\mathbf{f}_B = \mathbf{V}_{01}\,\underline{\mathbf{f}_{A1}}(1) + s\mathbf{V}_{11}\,\underline{\mathbf{f}_{A1}}(1) \tag{28}$$

with definitions

$$[\mathbf{V}_{01}^{\mathrm{T}} \quad \mathbf{V}_{11}^{\mathrm{T}}] = [\mathbf{I} \quad \mathbf{0}]\,\underline{\mathbf{T}_L}. \tag{29}$$

5. Independent modal control

To facilitate true independent modal control the modal equations of motion must be decoupled both externally and internally [11]. It has so far been shown how to decouple the unforced equations of motion but the diagonalised system matrices remain coupled by the control forces unless the controller is designed independently such that the controller matrix remains decoupled. In practice this means that the force controller must be designed in the modal space. One can thus define the modal equations of motion as

$$\mathbf{M}_B\ddot{\mathbf{q}}_m + \mathbf{D}_B\dot{\mathbf{q}}_m + \mathbf{K}_B\mathbf{q}_m = \mathbf{f}_{\mathrm{mod}} \tag{30}$$

with $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B \in \mathbb{R}^{n\times n}$ the diagonal modal system matrices and $\mathbf{q}_m \in \mathbb{R}^n$ the modal coordinates.

Eq. (30) represents n sdof systems corresponding to each mode of vibration. It is possible to use proportional-derivative control to directly affect the modal stiffness and damping properties of these modes. A controller of this form is introduced

$$\mathbf{f}_{\mathrm{mod}} = \mathbf{G}_k\mathbf{q}_m + \mathbf{G}_d\dot{\mathbf{q}}_m, \tag{31}$$

$\mathbf{G}_k$ and $\mathbf{G}_d$ represent the diagonal modal stiffness and damping gains matrices. Direct addition to the modal damping and stiffness matrices represents direct pole placement and has the advantage of being able to directly affect the poles of the system.

In general as many modes can be controlled as actuators available. As previously stated for the purpose of this paper the number of actuators is set to the number of modelled modes without any loss of generality. For conventional second-order control the modal force can be typically converted back into the physical domain fairly easily, as illustrated by Baz et al. [6]. For the SPT approach the left filter has already been defined, and one can see that the physical and modal forces are related by the relationship

$$\mathbf{f}_{\mathrm{mod}} = \mathbf{V}_{01}\mathbf{f}_{\mathrm{phy}} + \mathbf{V}_{11}\dot{\mathbf{f}}_{\mathrm{phy}}. \tag{32}$$

One can rearrange Eq. (32) to give the physical force in regards to the modal force

$$\dot{\mathbf{f}}_{\mathrm{phy}} = \mathbf{V}_{11}^{-1}(\mathbf{f}_{\mathrm{mod}} - \mathbf{V}_{01}\mathbf{f}_{\mathrm{phy}}). \tag{33}$$

Since the modal filter illustrated by Eq. (33) represents a first-order filter, a necessary requirement is for the real components of eigenvalues $\mathbf{V}_{11}^{-1}\mathbf{V}_{01} > 0$ for the filter to be stable. The stability of the filter is discussed later in this paper.

6. Numerical Example 1

Consider the deliberate non-classically damped second-order system with matrices

$$\mathbf{K}_A = \text{diag}\begin{bmatrix} 50 \\ 70 \\ 90 \\ 10 \end{bmatrix}, \quad \mathbf{D}_A = \begin{bmatrix} 11 & -2 & 0 & 3 \\ -2 & 16 & 5 & -1 \\ 0 & 5 & 11 & 2 \\ 3 & -1 & 2 & 14 \end{bmatrix}, \quad \mathbf{M}_A = \text{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{34}$$

subjected to initial displacements $\mathbf{q}(0) = [3\ 9\ 0\ 0]^{\mathrm{T}}$ and zero initial velocities. Forces may be applied at all locations and all of the physical displacements are available for observation.

It is decided for illustrative purposes to remove the modal damping of the third mode and set the natural frequency of this mode to 10 Hz. The SPT method is compared directly with the conventional IMSC method and conclusions drawn. For the respective methods one finds the modal controller matrices to be

$$\mathbf{f}_{\mathrm{m3\_spt}} = -14.434\,\dot{\mathbf{q}}_{\mathrm{m3\_spt}} + 3937.7\,\mathbf{q}_{\mathrm{m3\_spt}}, \tag{35}$$

$$\mathbf{f}_{\mathrm{m3\_imsc}} = -16\,\dot{\mathbf{q}}_{\mathrm{m3\_imsc}} + 3877.8\,\mathbf{q}_{\mathrm{m3\_imsc}}. \tag{36}$$

Figs. 1 and 2 illustrate the physical and modal displacements of the system when the SPT-modal controller is off and on, respectively. As may be observed from the comparison between the modal responses in Figs. 1 and 2, the third mode is successfully made undamped with a natural frequency of 10 Hz. The modes are completely decoupled and the controller only affects mode 3 and leaves the remaining modes unaltered. The

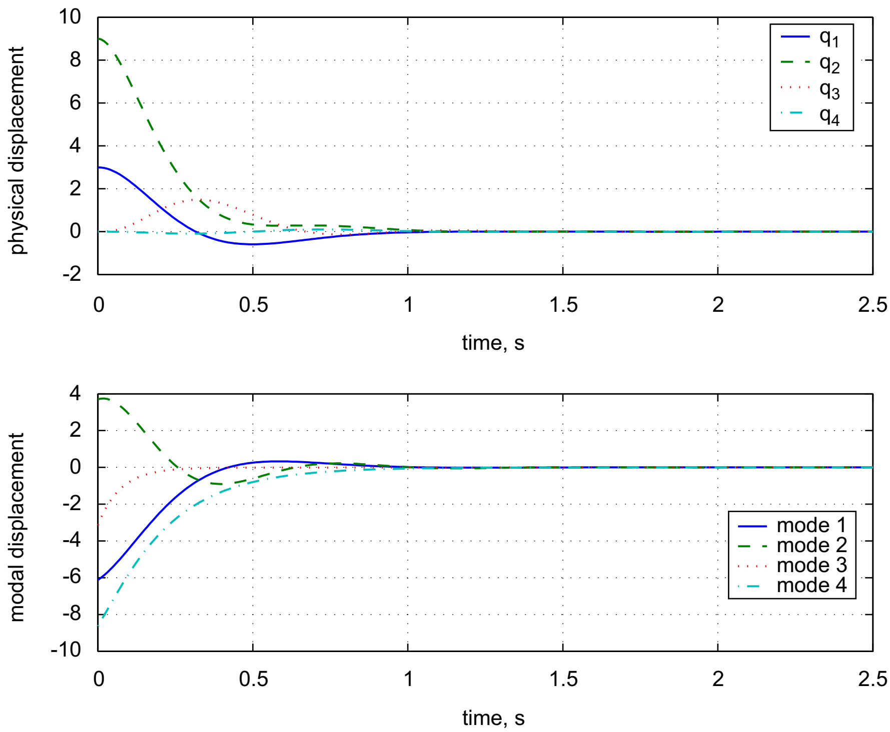

Fig. 1. Numerical example 1: Uncontrolled physical and modal responses, SPT method.

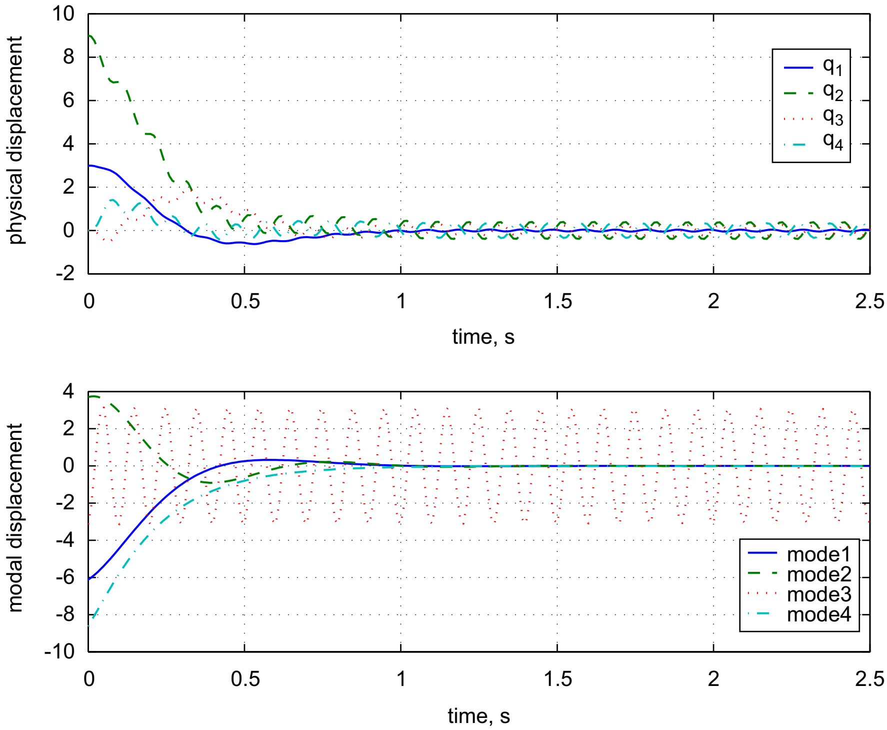

Fig. 2. Numerical example 1: Controlled physical and modal responses, SPT method.

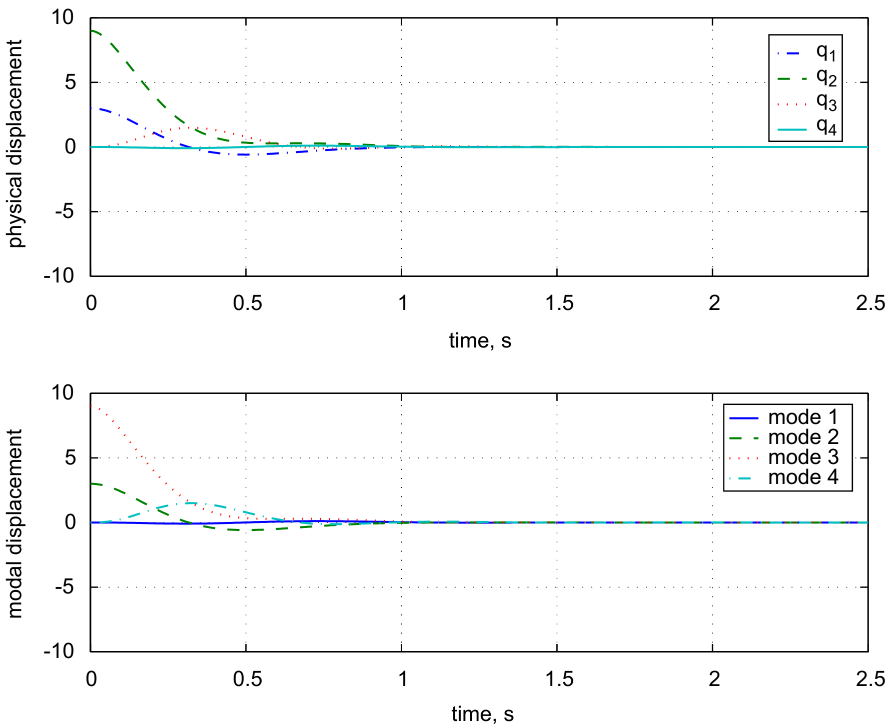

Fig. 3. Numerical example 1: Uncontrolled physical and modal responses, IMSC method.

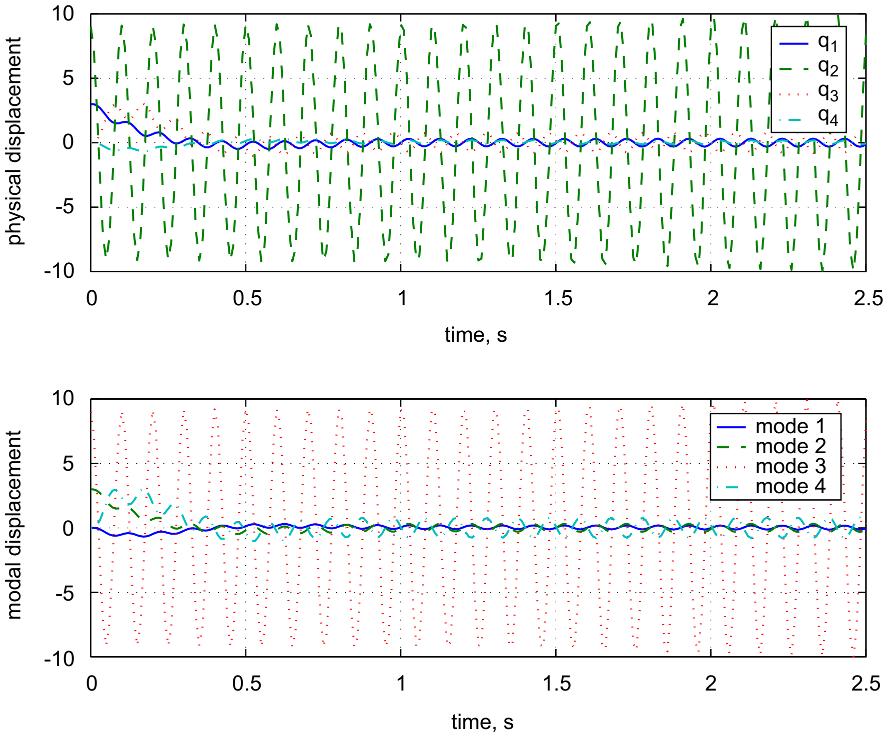

Fig. 4. Numerical example 1: Controlled physical and modal responses, IMSC method.

physical effect of the controller may be observed from the physical displacements illustrated in these figures and the obvious effect is to make the system borderline stable, i.e. neither stable nor unstable.

Figs. 3 and 4 illustrates the IMSC controller applied to the non-classically damped system. As may be observed the third mode is again made undamped with a natural frequency of 10 Hz. However, the modal responses of the other modes is also affected. This is due to the coupling in the damping matrix which cannot be made diagonal using the undamped eigenvectors of the system. Thus, the IMSC method does not allow true decoupling of the system matrices.

Although this numerical example is simple it illustrates the advantage of the SPT method over the conventional IMSC method due to the fact that all three system matrices may be decoupled regardless of the structure of the damping. One may also observe that the IMSC method results in different modeshapes. The modeshapes of the SPT method no longer match the undamped modeshapes of the system as the IMSC modes do. However, the SPT-modes do represent physically meaningful quantities and this is observed immediately in the physical response of the system in Fig. 2 when the SPT controller is on.

7. Numerical Example 2

As a numerical example, a finite element model of a rotor-disc system is considered with four degrees of freedom at each node (2 translational, 2 rotational). The rotor-disc system is illustrated in Fig. 5.

The system is constructed from steel with Young's modulus, $E = 200\,\text{GPa}$ and density $\rho = 7800\,\text{kg/m}^3$. The model is split into 13 equal-length elements of 0.1 m and the discs have dimensions given in Table 1.

The bearings at each end of the rotor system are deliberately isotropic with stiffness and damping properties given in Table 2.

The constraint $r = n$ is now relaxed so that control forces are applied at node 8 in the x- and y-directions and similarly the displacements in the x-direction at this node are observed. For computational ease Guyan reduction [12] is used to reduce the model to 6 degrees of freedom corresponding to the x and y co-ordinates at

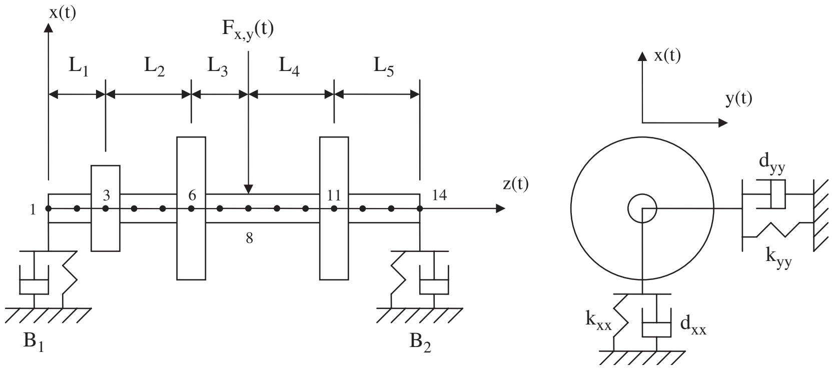

Fig. 5. Example 2: Rotor-disc system.

Table 1
Rotor example disc properties

Disc	Disc 1	Disc 2	Disc 3
Node	3	6	11
Thickness (m)	0.05	0.05	0.06
Inner diameter (m)	0.10	0.10	0.10
Outer diameter (m)	0.24	0.40	0.40

Table 2
Rotor example bearing stiffness and damping properties

Bearing	Bearing 1	Bearing 2
Stiffness K_{xx} (MN/m)	50	50
Stiffness K_{yy} (MN/m)	70	70
Stiffness D_{xx} (N/m/s)	500	500
Stiffness D_{yy} (N/m/s)	700	700

the disc locations. The system is operated at 2500 rpm and the uncontrolled system response is illustrated in Fig. 6.

Modal control dictates that each actuator controls an individual mode of vibration resulting in the number of modes to be controlled the same as the number of actuators available. The model allows for 2 modes to be controlled. It is decided to control the first two modes of vibration since these dominate the system response.

The sdof systems corresponding to the first two modes in modal space are

$$\ddot{\mathbf{q}}_{m1} + 0.37850\,\dot{\mathbf{q}}_{m1} + 1.4467 \times 10^5 \mathbf{q}_{m1} = \mathbf{f}_{m1}, \tag{37}$$

$$\ddot{\mathbf{q}}_{m2} + 0.32708\,\dot{\mathbf{q}}_{m2} + 1.5772 \times 10^5 \mathbf{q}_{m2} = \mathbf{f}_{m2}. \tag{38}$$

Optimal control is used to minimise the modal kinetic and potential energies such that controller gains are

$$\mathbf{G}_k = \begin{bmatrix} 4.999913 & 0 & 0 & \cdots & 0 \\ 0 & 4.999921 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{G}_d = \begin{bmatrix} 4.1096 & 0 & 0 & \cdots & 0 \\ 0 & 4.1570 & 0 & \cdots & 0 \end{bmatrix}. \tag{39}$$

The response of the system with the controller on is illustrated in Fig. 7. As expected the response of the system decays much faster than that for the uncontrolled system with the displacement converging to zero much more rapidly. This is due to targeting the first two modes of vibration of the system which dominate the system response. The modal control technique is indeed successfully applied to bring the system under control.

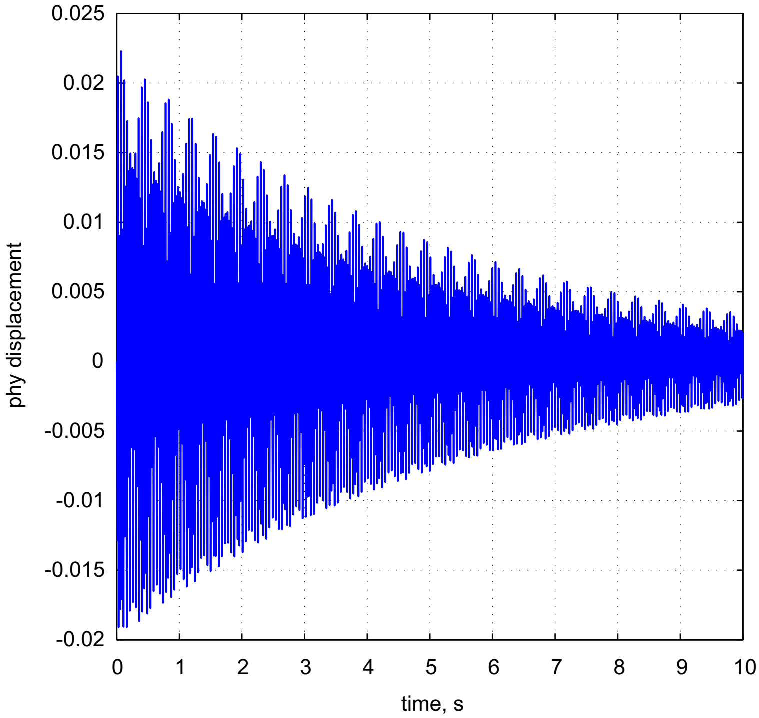

Fig. 6. Example 2: Uncontrolled physical displacements to initial conditions.

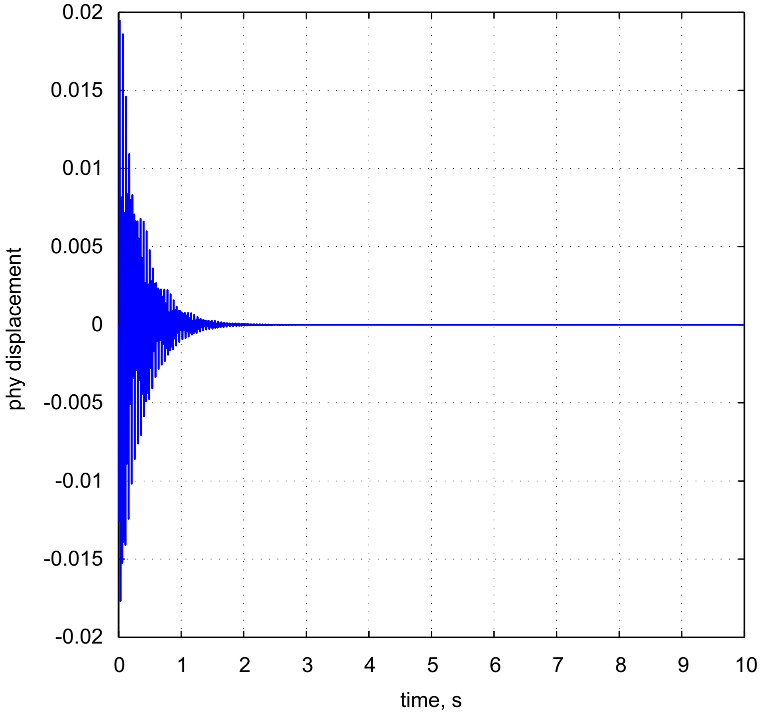

Fig. 7. Example 2: SPT-modal controlled physical displacements to initial conditions.

8. Reflexive SPTs and stable filters

Stable filters are defined when the eigenvalues of Eq. (33) have all non-negative real parts. At this stage it is appropriate to point out the non-uniqueness of the diagonalising SPT. It is possible to define an SPT for a sdof system which maps the system directly back onto itself. This means that the SPT is reflexive.

Establishing the LAMs for a sdof system

$$\underline{\mathbf{a}_0} = \begin{bmatrix} 0 & k \\ k & d \end{bmatrix}, \quad \underline{\mathbf{a}_1} = \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}, \quad \underline{\mathbf{a}_2} = \begin{bmatrix} -d & -m \\ -m & 0 \end{bmatrix}. \tag{40}$$

The SPTs for the sdof system may be defined as

$$\underline{\mathbf{t}_R} = \begin{bmatrix} f - \frac{1}{2}gd & -gm \\ gk & f + \frac{1}{2}gd \end{bmatrix}, \quad \underline{\mathbf{t}_L} = \begin{bmatrix} f + \frac{1}{2}gd & gm \\ -gk & f - \frac{1}{2}gd \end{bmatrix}, \tag{41}$$

where f and g are arbitrary scalars. By ensuring that the determinants of $\underline{\mathbf{t}_L}$ and $\underline{\mathbf{t}_R}$ are equal to 1 the reflexive SPT maps the sdof back onto itself completely ensuring the same sdof system is obtained and not a scalar multiple of itself.

Utilising the reflexive SPTs for the sdof systems in numerical experiments suggest that it is always possible to find a stable filter. This remains to be proved formally but the authors are content with the results from numerical trials. It is shown in Numerical Example 3 how to stabilise the filter matrices for a system which provides an initially unstable filter.

9. Numerical Example 3

The use of the reflexive SPTs in the search for stable modal filters is now demonstrated. For ease of illustration an arbitrary symmetric system is generated. Taking the arbitrary symmetric matrices to be

$$\mathbf{M}_A = \text{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{D}_A = \begin{bmatrix} 36 & 37 & 34 & 0 \\ 37 & 53 & 50 & 6 \\ 34 & 50 & 52 & 12 \\ 0 & 6 & 12 & 9 \end{bmatrix}, \quad \mathbf{K}_A = \begin{bmatrix} 91 & 75 & 55 & 69 \\ 75 & 68 & 60 & 52 \\ 55 & 60 & 154 & 109 \\ 69 & 52 & 109 & 186 \end{bmatrix}. \tag{42}$$

Since the system matrices are symmetric the resulting left and right SPT matrices are symmetric. Thus from Eq. (8) one may report the SPT construction matrices to be

$$\mathbf{F}_R = \mathbf{F}_L = \begin{bmatrix} -0.72015 & -0.46656 & 0.76672 & -0.51586 \\ 0.34606 & -0.17101 & 0.77114 & 1.2528 \\ -0.25642 & 1.2115 & 0.54219 & 0.059283 \\ -0.024509 & 0.054717 & 0.10386 & 0.051386 \end{bmatrix}, \tag{43}$$

$$\mathbf{G}_R = \mathbf{G}_L = \begin{bmatrix} 0.042865 & -0.05958 & 0.029348 & 0.037665 \\ 0.070925 & -0.014486 & -0.056858 & 0.013429 \\ -0.051694 & -0.03146 & 0.041829 & 0.036735 \\ 0.022864 & -0.0049255 & 0.012063 & -0.0082306 \end{bmatrix}. \tag{44}$$

Generating the $\mathbf{V}_{01}$ and $\mathbf{V}_{11}$ filter matrices using Eq. (29) it is apparent that the real part of the eigenvalues of $\left(\mathbf{V}_{11}^{-1}\mathbf{V}_{01}\right)$ are not all positive. Thus the filter is unstable. Utilising the (2 × 2) reflexive SPT applied to the first mode of the system allows the eigenvalues of the filter to be moved. Defining $f = \sin(\alpha)$ and $g = \cos(\alpha)$ where $0 \leqslant \alpha \leqslant 2\pi$. The variation of the minimum component of the real eigenvalue versus angle α is illustrated in Fig. 8. By varying α the eigenvalues of the filter changes. Thus, the reflexive SPT can be used to move the

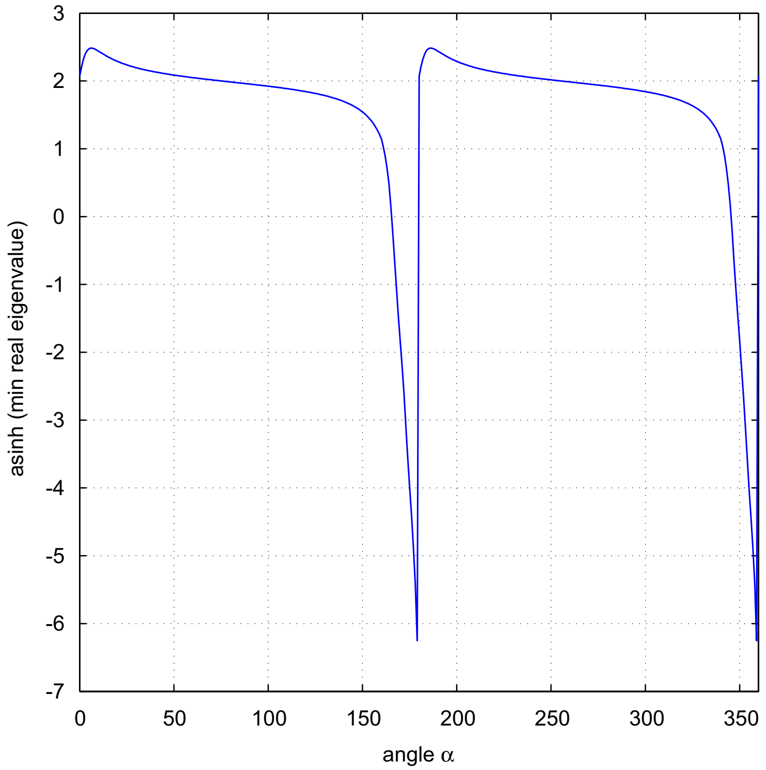

Fig. 8. Variation of angle α versus $\sinh^{-1}$ (min real eigenvalue).

eigenvalues of the filter into a stable region. Indeed n reflexive SPTs can be applied simultaneously to span a possible n-dimensional nonlinear space containing the stable filters.

10. Conclusions

In this paper, a novel modal control method has been presented which can be applied to non-classically damped systems. The method has been demonstrated through numerical examples and it has been illustrated that individual modes can be controlled and stable filters found numerically through the non-uniqueness of the SPTs.

The premise of this paper is to introduce possible new methods into the area of rotating machinery where skew-symmetry and gyroscopic coupling are regularly found in the system damping matrices. The method requires that no information be destroyed unlike conventional techniques which require that skew-symmetry be ignored for the modal control techniques to be usable.

Usually, systems require reduction in size due to numerical considerations. Traditional Guyan reduction models do not take into account damping properties. Alternative methods such as balanced truncation [7], traditionally place the system into state-space form before reduction, thus destroying the second-order properties of the system. Few methods have been developed to reduce the models in size for second-order systems. It would thus be beneficial to develop second-order model reduction methods that take into account damping whilst preserving second-order form.

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